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The higher order corrections to the equations that describe nonlinear wave motion in shallow water are derived from the water wave equations. In particular, the extended cylindrical Korteweg-de Vries and Kadomtsev-Petviashvili equations –which include higher order nonlinear, dispersive and nonlocal terms– are derived from the Euler system in (2+1) dimensions, using asymptotic expansions. It is thus found that the nonlocal terms are inherent only to the higher dimensional problem, both in cylindrical and Cartesian geometry. Asymptotic theory is used to study the resonant radiation generated by solitary waves governed by the extended equations, with an excellent comparison obtained between the theoretical predictions, for the resonant radiation amplitude, and the numerical solutions. In addition, resonant dispersive shock waves (undular bores) governed by the extended equations are studied. It is shown that the asymptotic theory, applicable for solitary waves, also provides an accurate estimate of the resonant radiation amplitude, of the undular bore.

Keywords: Solitary waves, undular bores, resonant radiation, asymptotic theory, extended Korteweg-de Vries equation, extended Kadomtsev-Petviashvili equation.

I. INTRODUCTION

The study of waves on the surface of a fluid is a classical topic in fluid mechanics, with a mathematical history dating back to the pioneering work of G.G. Stokes [1, 2], and summarised in the classic text of H. Lamb [3]. In fact, water wave theory forms the backbone of much of oceanography, ocean engineering and water engineering. While a classical problem, water waves and the solutions of the water wave equations are an on-going topic of research. The water wave equations are a nonlinear free surface problem consisting of a linear equation, namely Laplace's equation for the motion of the bulk fluid, together with nonlinear kinematic and dynamic boundary conditions which give continuity of the surface and continuity of pressure across the surface, respectively [4, 5]. It is these nonlinear boundary conditions which mean that the full water wave equations cannot be solved, in general [4]. For this reason, the water wave equations have been studied in various asymptotic regimes, including the linear and weakly nonlinear limits, leading to Stokes' expansions and studies of modulational instability [4]. Another widely studied asymptotic regime is the weakly nonlinear, long wave regime, for which the wavelength of the wave is much larger than the depth of the fluid, and the amplitude of the wave is much less than the fluid depth. This leads to equations of Boussinesq and Korteweg-de Vries (KdV) type when dispersion and nonlinearity are balanced [4, 5]. An additional attraction of this asymptotic limit is the integrable nature of the KdV equation [4, 6] and the applicability of the KdV equation to waves in nature in both the ocean and atmosphere [4, 5, 7, 8], as well as to waves in plasmas [9, 10], optical fibers [11, 12], Bose-Einstein condensates [13], nematic liquid crystals [14], exciton-polariton superfluids [15], and so on.

As stated, the KdV equation can be derived from the water wave equations in the long wavelength, weakly nonlinear limit using an asymptotic expansion in two small non-dimensional parameters, the scaled wave height and the inverse scaled wavelength. Extending this asymptotic expansion to the next order results in the extended Korteweg-de Vries (eKdV) equation

$$u_t + \frac{3}{2}uu_x + \frac{1}{6}u_{xxx} + \varepsilon\left(c_1u^2u_x + c_2u_xu_{xx} + c_3uu_{xxx} + c_4u_{xxxxx}\right) = 0, \quad (1)$$

where subscripts denote partial derivatives and ε is the scaled wave height parameter [16]. This eKdV equation arises in a number of physical contexts. It can be derived from the water wave equations for surface gravity waves, in which case the coefficient values are $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 5/12$ and $c_4 = 19/360$. The full eKdV equation (1) has been used to study higher order dispersive shock waves, termed undular bores in fluid mechanics [17] and solid mechanics [18], transcritical flow over topography [19], as well as solitary waves in weakly nonlocal media [20]. In the special case $c_2 = c_3 = c_4 = 0$, so that higher order nonlinearity dominates over higher order dispersion, the eKdV equation (1) is the Gardner equation, which is integrable; this equation arises for large amplitude internal water (ocean) waves [21–23], as well as in plasma physics [24, 25] and quantum fluid mechanics [26]. As well as studying higher order KdV-type solitary waves, to be discussed next, the Gardner equation has been used to study higher order dispersive shock waves, termed undular bores in fluid mechanics, [27] and their application to transcritical flow over topography [16, 28]. For $c_1 = c_2 = c_3 = 0$ the eKdV equation (1) reduces to the Kawahara equation [29]

$$u_t + \frac{3}{2}uu_x + \frac{1}{6}u_{xxx} + \varepsilon c_4 u_{xxxxx} = 0. \qquad (2)$$

This equation arises for gravity-capillary waves when the Bond number is near 1/3 [30]. As well as studying capillary waves, the Kawahara equation has been used to study resonant undular bores in nonlinear optics [31, 32].

The introduction of the higher order nonlinear, dispersive and nonlinear/dispersive terms to the KdV equation leads to genuinely new effects, not just small corrections. The solitary wave solution of the Kawahara equation (2) is resonant if $c_4 > 0$ in that linear dispersive radiation's phase velocity can match the solitary wave velocity, resulting in the solitary wave radiating and decaying [33–37]. As an undular bore is a modulated wavetrain, in its standard form with solitary waves at one edge and linear waves at the other [38], undular bores governed by the Kawahara equation (2) are also resonant, with the bore shedding a resonant wavetrain ahead of it [30]. Since the undular bore is formed from an initial step which connects two distinct levels, one of which is non-zero, this resonance does not cause the bore to decay, but results in non-standard forms if the initial step is large enough, which can include the near total destruction of the bore structure itself, with only a strong resonant wavetrain remaining [30, 39]. In addition to water wave theory, the Kawahara equation (2) has been shown to apply to the nonlinear optics of nematic liquid crystals [31, 40], so that resonant undular bores can exist in this medium as well [31, 32, 41].

Given this importance and widespread use of the eKdV equation in the one dimensional (1D) setting, in this work the extended cylindrical Korteweg-de Vries (ecKdV) and the extended Kadomtsev-Petviashvili (eKP) equations will be derived from the full water wave equations in the quasi-1D and two-dimensional (2D) settings, respectively. The KP equation is the 2D equivalent of the KdV equation when weak lateral dispersion is included [42, 43], while the cKdV equation is the radially symmetric two space dimensional equivalent of the KdV equation [44]. It will be found that as well as the inclusion of the fifth derivative term u_{xxxxx}, as for the eKdV equation— which can lead to the resonance discussed above— the ecKdV and eKP equations include nonlocal, integral-type terms, which lead to qualitatively different behaviour to the cKdV and KP equations. These extended equations will be used to study solitary wave resonance due to the fifth derivative. As noted above, solitary wave solutions of the Kawahara equation (2) are in resonance with dispersive radiation due to the fifth derivative u_{xxxxx} term leading to non-convex dispersion for $c_4 > 0$ [34]. While the full water wave eKdV equation (1) with water wave coefficients c_i, $i = 1, \ldots, 4$, has a fifth derivative term of the appropriate sign to lead to resonance between the linear wave phase velocity and the solitary wave velocity, such a resonance has not been observed in numerical solutions [19]. In the case of the eKdV equation an asymptotic study will show that there is a node in the resonant wave amplitude for certain combinations of c_i, $i = 1, \ldots, 4$. It is found that the water wave coefficients nearly satisfy one of these nodal relations. The existence of this resonant wave amplitude node in higher dimensions is investigated using the eKdV and ecKdV equations derived in this work. This study of the dependence of the resonant radiation on the higher order coefficients is extended to resonant undular bores governed by the extended KdV and cKdV equations, with resonant wave amplitude minima found for the water wave coefficients, as for resonant solitary waves.

Although the above analysis and results refer to the shallow water wave problem, we also show that a connection with other physical contexts is also possible. In particular, we employ an asymptotic expansion method – similar to the one used to treat the Euler system– and reduce a generic nonlocal 2D nonlinear Schrödinger (NLS) model that governs beam propagation in media featuring a spatial nonlocal nonlinearity [45] (such as nematic liquid crystals [46, 47]) to the ecKdV equation. The latter has a form similar to the one which was derived for shallow water waves, which suggests that phenomena that occur in shallow water may also occur in optical systems.

Our presentation is organized as follows. In Section II, we present the framework of the Euler (or water wave) equations, while in Sections III and IV, we derive the ecKdV and the eKP equations, respectively. In Sections V and VI we analyze solitary wave and undular bore resonance, respectively, for both the 1D and quasi-1D (polar coordinate) setting. In Section VII, we present the derivation of the ecKdV equation from a nonlocal NLS model. Finally, in Section VIII, we summarize our conclusions.

II. WATER WAVE EQUATIONS

Let us consider gravity waves on the surface of an incompressible, inviscid fluid of undisturbed depth h. The fluid velocity $\boldsymbol{u}$ can then be expressed in terms of the velocity potential ϕ as $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. The x and y coordinates are taken in the horizontal plane with the z direction vertically upwards, opposite to the direction of gravity, the acceleration due to gravity being denoted by g. The displacement of the fluid surface from the undisturbed state is taken as $z = \eta(x, y, t)$, so that $z = 0$ is the undisturbed level. The water wave equations are then set in non-dimensional form, with the z coordinate scaled by the depth h, x and y by typical wavelengths λ_x and λ_y in these directions, respectively, time t by $\lambda_x/\sqrt{gh}$, the surface displacement η by a typical wave amplitude a and the velocity potential is measured in units of $\lambda_x ga/\sqrt{gh}$. It is noted that $c_0 = \sqrt{gh}$ is the linear long wave speed. The dimensionless water wave equations for surface gravity waves are then [4, 5]

$$\phi_{zz} + \mu^2\phi_{xx} + \mu^2\delta^2\phi_{yy} = 0, \quad -1 < z < \varepsilon\eta, \qquad (3)$$

which is Laplace's equation, in the fluid bulk, together with the impenetrable boundary condition:

$$\phi_z = 0, \quad z = -1, \tag{4}$$

at the fluid bottom and the dynamic and kinematic boundary conditions

$$\phi_t + \frac{\varepsilon}{2}\left(\phi_x^2 + \delta^2\phi_y^2 + \frac{1}{\mu^2}\phi_z^2\right) + \eta = 0, \quad z = \varepsilon\eta, \tag{5}$$

$$\mu^2\left[\eta_t + \varepsilon\left(\phi_x\eta_x + \delta^2\phi_y\eta_y\right)\right] = \phi_z, \quad z = \varepsilon\eta, \tag{6}$$

In these non-dimensional water wave equations, the non-dimensional wave parameters are $\varepsilon = a/h$ for the wave amplitude, $\delta = \lambda_x/\lambda_y$ for the ratio of the wavelengths in the x and y directions and $\mu = h/\lambda_x$ for the wavelength, that is dispersion. In the present work, we consider weakly nonlinear long waves, that is the wavelength is much greater than the water depth, i.e., $\mu \ll 1$, and the wave amplitude is much less than the fluid depth, so that $\varepsilon \ll 1$. The usual KdV-type balance between weakly dispersive and weakly nonlinear effects will be used with $\varepsilon = \mu^2$ [4].

III. EXTENDED CKDV EQUATION

Let us now consider the water wave equations (3)–(6) in the weakly nonlinear, long wave limit for the special case of quasi-1D circularly symmetric waves, leading to the cKdV equation [44], but extended to the next order in the asymptotic expansion, leading to the ecKdV equation. In this case, taking $\delta = 1$, the water wave equations in plane polar coordinates read:

$$\phi_{zz} + \varepsilon\left(\phi_{rr} + \frac{1}{r}\phi_r\right) = 0, \tag{7}$$

in the fluid bulk, together with the boundary conditions

$$\phi_z = 0 \quad \text{at} \quad z = -1, \tag{8a}$$

$$\phi_t + \frac{\varepsilon}{2}\left(\phi_r^2 + \frac{1}{\varepsilon}\phi_z^2\right) + \eta = 0 \quad \text{at} \quad z = \varepsilon\eta, \tag{8b}$$

$$\eta_t + \varepsilon\phi_r\eta_r = \frac{1}{\varepsilon}\phi_z \quad \text{at} \quad z = \varepsilon\eta. \tag{8c}$$

Following the derivation of the standard cKdV equation [7] (see also [48]), we introduce the stretched radial R and time T variables and scaled velocity potential Φ and surface displacement H:

$$R = \varepsilon(r - t), \quad T = \varepsilon^4 t, \quad \phi = \varepsilon\Phi, \quad \eta = \varepsilon^2 H. \tag{9}$$

We now asymptotically expand the velocity potential Φ as follows:

$$\Phi = \Phi_0 + \varepsilon^3\Phi_1 + \varepsilon^6\Phi_2 + \varepsilon^9\Phi_3 + \cdots. \tag{10}$$

Laplace's equation for the fluid, Eq. (7), then becomes:

$$\begin{aligned}&(R + T/\varepsilon^3)\Phi_{0zz} + T(\Phi_{1zz} + \Phi_{0RR})\\ &+\varepsilon^3\left[R\Phi_{1zz} + T\Phi_{2zz} + (R\Phi_{0R})_R + T\Phi_{1RR}\right]\\ &+\varepsilon^6\left[R\Phi_{2zz} + T\Phi_{3zz} + (R\Phi_{1R})_R + T\Phi_{2RR}\right] = O(\varepsilon^9).\end{aligned}$$

Solving this equation at each order of ε, and applying the bottom boundary condition (8a), gives

$$\begin{aligned}\Phi_0 &= A(R,T), \quad \Phi_1 = -\frac{(z+1)^2}{2}A_{RR},\\ \Phi_2 &= -\frac{(z+1)^2}{2T}A_R + \frac{(z+1)^4}{24}A_{RRRR},\\ \Phi_3 &= \frac{(z+1)^2 R}{2T^2}A_R + \frac{(z+1)^4}{12T}A_{RRR}\\ &- \frac{(z+1)^6}{720}A_{RRRRRR},\end{aligned} \tag{11}$$

where any homogeneous solutions that arise in higher-order terms are absorbed into the leading order solution Φ_0. These solutions are then substituted into the surface boundary conditions (8b) and (8c), keeping terms up to $O(\varepsilon^6)$.

Differentiating the dynamic boundary condition (8b) with respect to R yields:

$$\begin{aligned}&H_R - w_R + \varepsilon^3\left(w_T + ww_R + \frac{1}{2}w_{RRR}\right)\\ &+\varepsilon^6\left(\frac{1}{2T}w_{RR} + H_R w_{RR} + \frac{1}{2}w_R w_{RR} - \frac{1}{2}w_{RRT}\right.\\ &\left.+Hw_{RRR} - \frac{1}{2}ww_{RRR} - \frac{1}{24}w_{RRRRR}\right) = 0,\end{aligned} \tag{12}$$

while the kinematic boundary condition (8c) yields:

$$\begin{aligned}-H_R + w_R &+ \varepsilon^3\left(\frac{1}{T}w + H_T + (Hw)_R - \frac{1}{6}w_{RRR}\right)\\ &+ \varepsilon^6\left(-\frac{R}{T^2}w + \frac{1}{T}Hw - \frac{1}{3T}w_{RR}\right.\\ &\left.- \frac{1}{2}(Hw_{RR})_R + \frac{1}{120}w_{RRRRR}\right) = 0,\end{aligned} \tag{13}$$

where we have introduced the new variable w with $A = w_R$. To make equations (12) and (13) consistent, we set

$$w = H + \varepsilon^3 w_1 + \varepsilon^6 w_2 + \ldots, \tag{14}$$

and retrieve from compatibility the functions w_1 and w_2:

$$w_1 = -\frac{1}{4}H^2 + \frac{1}{3}H_{RR} - \frac{1}{2T}(\partial_R^{-1}H), \tag{15a}$$

$$\begin{aligned}w_2 &= \frac{1}{8}H^3 + \frac{3}{16}H_R^2 + \frac{1}{2}HH_{RR} + \frac{1}{10}H_{RRRR}\\ &+ \frac{1}{T}\left[\frac{1}{6}H_R - \frac{1}{16}(\partial_R^{-1}H^2)\right]\\ &+ \frac{1}{T^2}\left[\frac{1}{2}(\partial_R^{-1}RH) + \frac{5}{8}(\partial_R^{-2}H)\right].\end{aligned} \tag{15b}$$

Here, the operator ∂_R^{-1} is defined as

$$\partial_R^{-1} H = \int_0^R H(R', T)\, dR'. \qquad (16)$$

Finally, substituting these expressions for w_1 and w_2, and hence w, into the kinematic boundary condition (12) gives the extended cylindrical KdV (ecKdV) equation:

$$\begin{aligned} H_T &+ \frac{3}{2} H H_R + \frac{1}{6} H_{RRR} + \frac{1}{2T} H + \varepsilon^3 \left(-\frac{3}{8} H^2 H_R + \frac{23}{24} H_R H_{RR} + \frac{5}{12} H H_{RRR} + \frac{19}{360} H_{RRRRR} \right) \\ &+ \frac{\varepsilon^3}{T} \left[\frac{3}{16} H^2 + \frac{1}{4} H_{RR} - \frac{1}{2} H_R \partial_R^{-1}(H) \right] + \frac{\varepsilon^3}{T^2} \left[-\frac{R}{2} H + \frac{1}{8} \partial_R^{-1}(H) \right] = 0. \end{aligned} \qquad (17)$$

Note that in the 1D case, i.e., ignoring the terms with $1/T$ and $1/T^2$ which result from the ϕ_r/r term in Laplace's equation, the above extended cKdV (ecKdV) equation reduces to the usual, 1D, extended KdV equation for surface gravity waves [16, 49]. It should also be mentioned that, while the radially symmetric water wave equations (8) have no dependence on the polar angle θ, it still describes radially symmetric water waves which are not purely1D objects.

We additionally note that the higher dimension has introduced terms in the ecKdV equation which are nonlocal due to the operator ∂_R^{-1}. However, we note that for large time T, the ecKdV equation reduces to the eKdV equation. This is expected as for large T the radius of curvature of a wave will be small as it propagates into large R, so that the wave is essentially one dimensional. This will become more apparent in the fully 2D case that we study below.

We conclude this Section with a comment on the connection between the ceKdV equation (17) with its Cartesian counterpart. We recall that, in the absence of higher-order effects, the "regular" cKdV and KdV equations can be linked to each other via a transformation [44, 50]. Such a transformation also exists for the ecKdV equation (17) that maps it to a perturbed KdV equation. In detail, upon defining:

$$H = \frac{R}{3T} - \frac{1}{2T} u(\xi, \tau) - \frac{4\varepsilon^3}{3} \xi^2 \tau^2 \log \tau, \qquad (18)$$

$$R = \frac{\xi}{\tau}, \quad T = -\frac{1}{2\tau^2}, \qquad (19)$$

we may transform the ecKdV equation (17) to the perturbed KdV equation

$$\begin{aligned} u_\tau &+ \frac{3}{2} u u_\xi + \frac{1}{6} u_{\xi\xi\xi} + \varepsilon^3 \left[-\frac{\xi}{6}(11 + 24 \log \tau) u - \frac{\xi^2}{2}(1 + 4 \log \tau) u_\xi - \frac{1}{6} \partial_\xi^{-1}(u) \right] + \varepsilon^3 \tau \left[-\frac{1}{8} u^2 + \frac{1}{2} \xi u u_\xi \right. \\ &\left. - \frac{41}{36} u_{\xi\xi} - \frac{5}{18} \xi u_{\xi\xi\xi} + u_\xi \partial_\xi^{-1}(u) \right] + \varepsilon^3 \tau^2 \left[-\frac{3}{8} u^2 u_\xi + \frac{23}{24} u_\xi u_{\xi\xi} + \frac{5}{12} u u_{\xi\xi\xi} + \frac{19}{360} u_{\xi\xi\xi\xi\xi} \right] = O(\varepsilon^6). \end{aligned} \qquad (20)$$

We note that the $O(\varepsilon^3)$ "correction" to the original transformation for H in (19) serves to cancel any inhomogeneous terms produced up to that order.

IV. THE EXTENDED KP EQUATION

Next, we derive the fully 2D extended KP equation as an approximation to the water wave equations in the weakly nonlinear, long wave limit for which there is weak lateral dispersion [42, 43]. As for the derivation of the ecKdV equation, the dimensional water wave equations for water waves propagating over a flat bottom consist of Laplace's equation for the fluid bulk:

$$\phi_{zz} + \varepsilon(\phi_{xx} + \varepsilon \delta^2 \phi_{yy}) = 0, \quad -1 < z < \varepsilon\eta, \qquad (21)$$

together with the bottom boundary condition,

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -1, \qquad (22)$$

and the dynamic and kinematic boundary conditions on the free surface,

$$\phi_t + \frac{1}{2}\varepsilon\left(\phi_x^2 + \varepsilon\delta^2\phi_y^2 + \frac{1}{\varepsilon}\phi_z^2\right) + \eta = 0 \quad \text{at} \quad z = \varepsilon\eta, \tag{23a}$$

$$\eta_t + \varepsilon\left(\phi_x\eta_x + \varepsilon\delta^2\phi_y\eta_y\right) = \frac{1}{\varepsilon}\phi_z \quad \text{at} \quad z = \varepsilon\eta. \tag{23b}$$

We note that, once again, the parameter δ^2, measuring the ratio of wavelengths in the x and y directions, is of order $O(1)$ and while it can be absorbed trivially via a change of coordinates, it is left to act as a measure of the dimensionality contribution.

Similarly to the derivation of the ecKdV equation, we expand the velocity potential ϕ as:

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \varepsilon^3\phi_3 + \cdots . \tag{24}$$

Substituting this expansion into Laplace's equation (21) gives:

$$\begin{aligned}\phi_{0zz} &+ \varepsilon(\phi_{1zz} + \phi_{0xx} + \delta^2\phi_{0yy}) \\ &+ \varepsilon^2(\phi_{2zz} + \phi_{1xx} + \delta^2\phi_{1yy}) \\ &+ \varepsilon^3(\phi_{3zz} + \phi_{2xx} + \delta^2\phi_{2yy}) = O(\varepsilon^4).\end{aligned} \tag{25}$$

We now solve Laplace's equation at each order of ε using the bottom boundary condition (22). Again, solving the differential equations at each order of ε and applying the bottom condition gives the solutions:

$$\begin{aligned}\phi_0(x,y,z,t) &= A(x,y,t), \\ \phi_1(x,y,z,t) &= -\frac{(z+1)^2}{2}(A_{xx} + \delta^2 A_{yy}), \\ \phi_2(x,y,z,t) &= \frac{(z+1)^4}{24}(A_{xxxx} + 2\delta^2 A_{xxyy} + \delta^4 A_{yyyy}), \\ \phi_3(x,y,z,t) &= \frac{(z+1)^6}{720}(A_{xxxxxx} + 3\delta^2 A_{xxxxyy} \\ &+ 3\delta^4 A_{xxyyyy} + \delta^6 A_{yyyyyy}).\end{aligned} \tag{26}$$

Differentiating the dynamic boundary condition (23a) with respect to x, substituting the solutions (26) of Laplace's equation and introducing $A = w_x$, casts Bernoulli's equation (23a) and the kinematic boundary condition (23b) into the forms:

$$\begin{aligned}w_t + \eta_x &+ \varepsilon\left(ww_x - \frac{1}{2}w_{xxt}\right) + \varepsilon^2\Bigg[\delta^2 w_y(\partial_x^{-1}w_y) \\ &- \frac{1}{2}\delta^2 w_{yyt} - \eta_x w_{xt} + \frac{1}{2}w_x w_{xx} - \eta w_{xxt} \\ &- \frac{1}{2}ww_{xxx} + \frac{1}{24}w_{xxxxt}\Bigg] = 0,\end{aligned} \tag{27}$$

and

$$\begin{aligned}\eta_t + w_x &+ \varepsilon\left[\delta^2(\partial_x^{-1}w_{yy}) + (\eta w)_x - \frac{1}{6}w_{xxx}\right] \\ &+ \varepsilon^2\Bigg[\delta^2\eta(\partial_x^{-1}w_{yy}) + \delta^2\eta_y(\partial_x^{-1}w_y) - \frac{1}{3}\delta^2 w_{xyy} \\ &- \frac{1}{2}(\eta w_{xx})_x + \frac{1}{120}w_{xxxxx}\Bigg] = 0.\end{aligned} \tag{28}$$

To make these two equations consistent we again set

$$w = \eta + \varepsilon(w_1 + \delta^2 w_{12}) + \varepsilon^2(w_2 + \delta^2 w_{21}) + \ldots, \tag{29}$$

where we have isolated different corrections to emphasize the role of the added dimensionality. Substituting this expansion into (27) and (28) gives:

$$w_1 = -\frac{1}{4}\eta^2 + \frac{1}{3}\eta_{xx}, \tag{30a}$$

$$w_{12} = -\frac{1}{2}\partial_x^{-2}\eta_{yy}, \tag{30b}$$

$$w_2 = \frac{1}{8}\eta^3 + \frac{3}{16}\eta_x^2 + \frac{1}{2}\eta\eta_{xx} + \frac{1}{10}\eta_{xxxx}, \tag{30c}$$

$$\begin{aligned}w_{21} &= \frac{1}{6}\eta_{yy} - \frac{3}{8}\partial_x^{-1}(\eta\partial_x^{-1}(\eta_{yy})) \\ &+ \frac{5}{8}\partial_x^{-2}(\eta\eta_{yy} + \eta_y^2) + \frac{3\delta^2}{8}\partial_x^{-4}(\eta_{yyyy}).\end{aligned} \tag{30d}$$

We note that w_1 and w_2 have already been found in Ref. [49]. Finally, substituting these w_{ij} into the kinematic boundary condition (28) gives the extended KP (eKP) equation:

$$\begin{aligned}&(\eta_t + \eta_x)_x + \varepsilon\left(\frac{3}{2}\eta\eta_x + \frac{1}{6}\eta_{xxx}\right)_x + \varepsilon\delta^2\left(\frac{1}{2}\eta_{yy}\right) + \varepsilon^2\left(-\frac{3}{8}\eta^2\eta_x + \frac{23}{24}\eta_x\eta_{xx} + \frac{5}{12}\eta\eta_{xxx} + \frac{19}{360}\eta_{xxxxx}\right)_x \\ &+ \varepsilon^2\delta^2\left[\frac{9}{8}\eta_y^2 + \frac{1}{4}\eta\eta_{yy} + \frac{1}{4}\eta_{xxyy} - \frac{1}{2}\eta_{xx}\partial_x^{-2}(\eta_{yy}) - \frac{3}{8}\eta_x\partial_x^{-1}(\eta_{yy}) + \eta_{xy}\partial_x^{-1}(\eta_y) - \frac{\delta^2}{8}\partial_x^{-2}(\eta_{yyyy})\right] = 0.\end{aligned} \tag{31}$$

It is clear that the effect of the higher dimensionality in this eKP equation is more pronounced than for the ecKdV equation (17). Indeed, the additional terms appearing over those in the ecKdV equation, measured by the parameter δ, are highly nonlocal due to the operator ∂_x^{-1}, which now occurs in multiple terms.

Notably, the eKP (31) and ecKdV (17) equations are related. Indeed, as the radius of the waves grows, or in the limit $T \gg 1$, the wave front becomes locally flat and the ecKdV equation (17) may be approximately described by the eKdV equation

$$\eta_t + \eta_r + \varepsilon\left(\frac{3}{2}\eta\eta_r + \frac{1}{6}\eta_{rrr}\right) + \varepsilon^2\left(-\frac{3}{8}\eta^2\eta_r + \frac{23}{24}\eta_r\eta_{rr} + \frac{5}{12}\eta\eta_{rrr} + \frac{19}{360}\eta_{rrrrr}\right) = 0,$$

where all variables are defined as in (9).

V. SOLITARY WAVE RESONANCE

A. One space dimension

Let us now consider the behaviour of solutions, particularly solitary wave solutions, of the eKdV equation (1) and the ecKdV equation (17). To connect directly with previous work on $(1+1)$ dimensional eKdV equations, we shall rescale the eKdV equation (1) to:

$$\begin{aligned} &v_\tau + 6vv_\xi + v_{\xi\xi\xi} \\ &+\varepsilon_1\left(d_1v^2v_\xi + d_2v_\xi v_{\xi\xi} + d_3vv_{\xi\xi\xi} + v_{\xi\xi\xi\xi\xi}\right) = 0, \end{aligned} \quad (32)$$

where we have used the scalings:

$$\begin{aligned} &\xi = x, \quad \tau = \frac{1}{6}t, \quad v = \frac{3}{2}u, \\ &\varepsilon_1 = 6\varepsilon c_4, \quad d_1 = \frac{4}{9}\frac{c_1}{c_4}, \quad d_2 = \frac{2}{3}\frac{c_2}{c_4}, \quad d_3 = \frac{2}{3}\frac{c_3}{c_4}. \end{aligned} \quad (33)$$

For the choices $d_1 = d_2 = d_3 = 0$ this equation is the Kawahara equation (2). It is well known that the Kawahara equation possesses solitary wave solutions in resonance with dispersive radiation as the linear dispersion relation $\omega = -k^3 + \varepsilon_1 k^5$ is non-convex if $\varepsilon_1 > 0$, so that the linear phase velocity can coincide with the solitary wave velocity [34–37]. However, in the general case with all the higher order terms present in the eKdV equation the amplitude of this resonant radiation depends markedly on the values of the higher order coefficients d_1, d_2 and d_3. In particular, if the higher order coefficients satisfy the relation

$$d = 90 + d_1 - 3d_2 - 6d_3 = 0, \quad (34)$$

then exact solitary wave solutions with no associated resonant radiation exist, see (22) in [33]. The relation (34) includes well known families of integrable higher-order equations such as the Lax hierarchy and the Sawada-Kotera equation. For the eKdV equation (32) the coefficients for water waves, see (1), are

$$d_1 = -\frac{60}{19}, \quad d_2 = \frac{230}{19}, \quad d_3 = \frac{100}{19}, \quad \varepsilon_1 = \varepsilon\frac{19}{60}. \quad (35)$$

While these coefficients (35) do not satisfy the node relation (34), numerical solutions of this equation for surface water wave undular bores show that the amplitude of the generated resonant radiation is very small [19]. The resonant radiation generated by solitary wave solutions of the eKdV equation will now be investigated using asymptotic theory. Resonant undular bores will then be studied, noting that undular bores are a modulated periodic wave with solitary waves at one edge. This analysis will verify the small resonant wave amplitude in the surface water wave case, both for solitary waves and undular bores, which was observed from numerical solutions.

Let us consider a travelling wave solution of the eKdV equation with $u = u(\theta)$, where $\theta = \xi - c\tau$. The eKdV equation then has the leading order asymptotic solitary wave solution:

$$v = 2\gamma^2\mathrm{sech}^2(\gamma\theta) + b\sin(k\theta + \phi), \quad (36)$$

$$b = -\frac{2\pi K}{\varepsilon_1}\exp\left(-\frac{\pi k}{2\gamma}\right), \quad c = 4\gamma^2 + \varepsilon_1 16\gamma^4, \quad (37)$$

for small $|\varepsilon_1|$, where b is the amplitude, k the wavenumber and ϕ the phase constant of the resonant wavetrain [34]. Note that at $O(1)$ this solitary wave is just the KdV soliton, with an attached small amplitude periodic wave of amplitude $O(b)$ at next order. The $O(\varepsilon_1)$ correction to the solitary wave itself is not needed for the current analysis. For resonance the phase velocity of the linear wavetrain is equal to the solitary wave velocity, giving $\omega/k = -k^2 + \varepsilon_1 k^4 = c$. Hence,

$$k = \varepsilon_1^{-1/2}(1 + 4\varepsilon_1\gamma^2)^{1/2}. \quad (38)$$

Now it can be seen from (36) that the amplitude b of the resonant radiation is exponentially small as $b \sim \varepsilon_1^{-1}\exp(\varepsilon_1^{-1/2})$ as $\varepsilon_1 \to 0$.

As the resonant tail amplitude is exponentially small, the techniques of exponential asymptotics must be used to obtain the amplitude of this tail [51]. To determine this resonant wavetrain, we rescale the eKdV equation (32) for the travelling wave solution $u = u(\theta)$ by

$$w = \varepsilon_1 v, \quad q = \varepsilon_1^{-1/2}\left[\theta - \frac{(2n+1)i\pi}{2\gamma}\right], \quad (39)$$

where the form of the spatial variable q is related to the structure of the poles of the soliton solution, see [33].

This transformation is made so that q is small near the poles of the soliton solution, near where the matching between the solitary wave and resonant radiation occurs. To leading order the eKdV equation then becomes:

$$6ww_q + w_{qqq} + d_1w^2w_q + d_2w_qw_{qq} + d_3ww_{qqq} + w_{qqqqq} = 0. \quad (40)$$

The solution to this inner problem for the resonant radiation must be found and then matched to that of the full problem with the solitary wave. The inner solution in the KdV case (no higher-order terms) is $w = -2/q^2$, which suggests a series solution of the form:

$$w = -\frac{2}{q^2} + \sum_{n=2}^{\infty} \frac{a_n}{q^{2n}}. \quad (41)$$

This series solution (41) is now substituted into the scaled eKdV equation (40), which then gives a recurrence relation for a_n. At lowest order we obtain $a_2 = -\frac{2}{9}d$, so that if the nodal relation (34) is satisfied, then a_2 and all the higher-order coefficients are zero and $K = 0$. In this case no radiation is generated and exact steady solitary wave solutions exist.

The coefficients a_n are calculated numerically from the recurrence relation for given choices of the d_i. The sequence of coefficients a_n is divergent, however. For large n it can be shown that the recurrence relation has the asymptotic solution $a_n = K(-1)^n(2n-1)!$. Dividing our numerically obtained coefficents a_n by the large n asymptotic solution gives a sequence of approximations for the constant K. Multiple applications (five to ten) of Aitken convergence acceleration (see [52]) are applied to the sequence for K in order to obtained a converged soution before round-off affects the result.

For the Kawahara equation, for which $d_1 = d_2 = d_3 = 0$, we find $K = 19.97$, which reproduces the result of [33]. For the water wave coefficients (35) we find $K = 0.768$, which is about 4% of the magnitude of K for the Kawahara case. This result explains the numerical observation of the small amplitude of the resonant radiation for surface water waves [19].

The eKdV equation (1) was solved numerically using the pseudo-spectral method of Fornberg and Whitham [53], as extended to enhance its stability at high wavenumbers through the use of an integrating factor to propagate the linear dispersion u_{xxx} and u_{xxxxx} in the eKdV equation (1) or H_{RRR} and H_{RRRRR} in the ecKdV equation (17)[54, 55]. This enhancement of stability is particularly important for the eKdV equation due to the fifth order dispersion. The nonlinear terms in these equations involving derivatives of u and H are calculated in Fourier space. The equations are then propagated forward in time t using the fourth order Runge-Kutta method. This propagation is done in Fourier space using the aforementioned integrating factor to propagate linear dispersion as this enhances the stability of the numerical scheme as the (high order) dispersion is propagated exactly [54, 55], in contrast to propagating in physical space for which very small time steps Δt are needed for stability due to the fifth order dispersion [53].

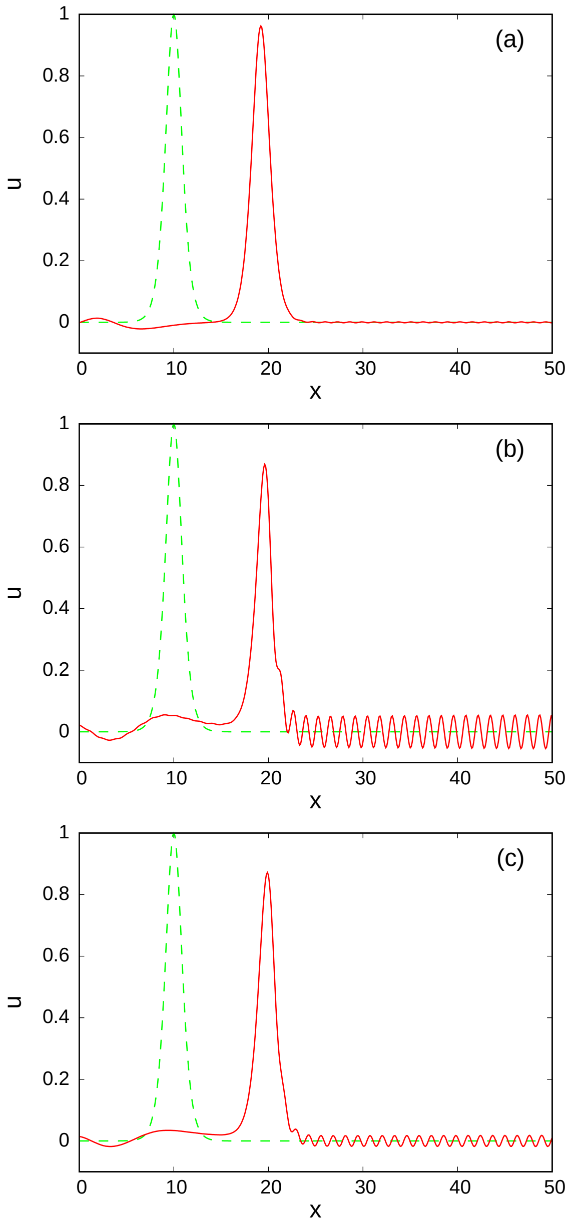

FIG. 1: Numerical solutions of the extended KdV equation (1). Green (dotted) line: KdV soliton initial condition at $t = 10$; red (full) line: solution at $t = 30$. (a) Water wave coefficients $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 5/12$ and $c_4 = 19/360$, (b) fifth order derivative only $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ and $c_4 = 19/360$, (c) higher order term uu_{xxx} vanishing, $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 0$ and $c_4 = 19/360$. Here, $a = 1$ for the KdV soliton (42) and $\varepsilon = 0.15$.

Figure 1 displays numerical solutions of the eKdV equation (1) for the KdV solitary wave initial condition

$$u = a\,\mathrm{sech}^2 \frac{\sqrt{3a}}{2}(x - 10). \quad (42)$$

The initial KdV soliton is centred at $x = 10$ as the corresponding variable to x for the ecKdV equation (17) is the polar radius R. Choosing the centre of the initial conditions the same for the eKdV and ecKdV equations allows direct comparisons between solutions of these two equations. For these comparisons, we use the scalings (33) which link the eKdV equation (32) used for the resonant radiation analysis back to the original eKdV equation, Eq. (1).

Many examples of the experimental generation, propagation and measurement of surface solitary water waves can be found in the literature. Lee *et al* [56] generated solitary waves with amplitude to depth ratios $\varepsilon = 0.11$, 0.19 and 0.29 and measured the wave profiles and particle velocities for each case. Hsu *et al* [57] undertook experiments to investigate the particle trajectories beneath a solitary water wave and considered wave amplitude to depth ratios in the range $\varepsilon = 0.182$ to 0.428. Hence, our choice of $\varepsilon = 0.15$ for the eKdV equation (1) and $\varepsilon^3 = 0.15$, $\varepsilon = 0.5313\ldots$, for the ecKdV equation (17) used in the numerical simulations of this paper is a reasonable choice within the lower end of the range of amplitudes used for experimental work on surface solitary waves. In order to have valid comparisons between the eKdV and ecKdV equations a high amplitude to depth ratio then needed to be chosen for the circularly symmetric equation.

Figure 1(a) shows the resonant wavetrain shed by the solitary wave for the higher order coefficients c_i, $i = 1, \ldots, 4$, taking the water wave values. The numerical tail amplitude is 1.5×10^{-3}, while the theoretical tail amplitude (36) is 1.6×10^{-3}. The comparison between the numerical and theoretical tail amplitudes is excellent and provides a theoretical explanation for why the tail amplitude is very small in the water wave case. Figure 1(b) displays a solution for the Kawahara equation (2). The resonant radiation shed by a solitary wave governed by the Kawahara equation has been well studied [34–37]. The numerical tail amplitude is 5×10^{-2}, while the theoretical amplitude is 4.1×10^{-2}, about 25 times larger than for the water wave case. The resonant radiation now has a significant amplitude, which results in a significant decrease in the amplitude of the solitary wave as the radiation is formed from the solitary wave itself. The critical dependence of the resonant wave amplitude on the exact values of the higher order coefficients c_i, $i = 1, \ldots, 4$, is further illustrated in Figure 1(c) for which only the coefficient of the higher order term $u_x u_{xx}$ vanishes. For this case $K = 6.05$ and the numerical tail amplitude is 1.8×10^{-2}, while the theoretical tail amplitude (36) is 1.3×10^{-2}. This is an intermediate case for which the amplitude of the resonant radiation is significantly reduced over that for the Kawahara equation case, but it is still not the trivial amplitude of the water wave eKdV equation.

B. Radially symmetric two space dimensions

As the ecKdV equation (17) is in the polar radial variable R, to enable the use of a pseudo-spectral method, the even extension of H into $R < 0$ was used in the pseudo-spectral numerical scheme used for the eKdV equation and outlined in the previous subsection. The extension of the results of the previous subsection for the eKdV equation will now be investigated for the ecKdV equation (17). Since as $T \to \infty$, the ecKdV equation (17) approaches the eKdV equation (1), it is expected that the resonant solitary wave solutions of these two equations should be similar, in particular, the dependence of the resonant radiation on the coefficients of the higher order terms. The water wave ecKdV equation (17) has the coefficient values $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 5/12$ and $c_4 = 19/360$.

This dependence of the amplitude of the resonant radiation generated by a solitary wave on the higher order coefficients is now examined in Fig. 2 for the ecKdV equation (17) using the same choices of coefficients as for the eKdV equation. This dependence is much less pronounced than for the eKdV equation, with little variation in the radiation amplitude as the higher order coefficients change. It should be noted that the amplitude of the solitary wave is decaying due to the $1/T$ terms in the ecKdV equation (17), as well as due to the shed radiation. In addition, the $H/(2T)$ term in the ecKdV equation (17) results in the amplitude of linear waves decaying as $(T - T_0)^{-1/2}$. Indeed, the shed radiation has amplitude 1.69×10^{-2} for the water wave coefficients, see Fig. 2(a), amplitude 2.48×10^{-2} for the Kawahara equation case of Fig. 2(b) and amplitude 1.33×10^{-2} for the ecKdV equation with only the higher order term HH_{RRR} vanishing, as shown in Fig. 2(c). In particular, there is no large reduction of the resonant radiation amplitude for the water wave coefficients as for one dimension. These results show that the $1/T$ terms of the ecKdV equation have a major effect on the initial evolution of the resonant radiation. However, incorporating the decay $(T-T_0)^{-1/2}$ into the theory for the eKdV equation of the previous subsection does not predict these resonant radiation amplitudes. The amplitude of the resonant radiation decays, so that the large amplitude resonant radiation for the Kawahara equation with $c_1 = c_2 = c_3 = 0$ in (1+1)D does not occur for the cylindrical Kawahara equation. In this context, it is noted that the ecKdV equation (17) approaches the eKdV equation as $T \to \infty$, so that in the long term the resonant radiation generated by the eKdV and ecKdV equations should converge. The analysis of this radiation is much more difficult than for the eKdV equation due to the lack of a suitable solitary wave solution of the cKdV equation on which to base this analysis and the time dependence of the solutions.

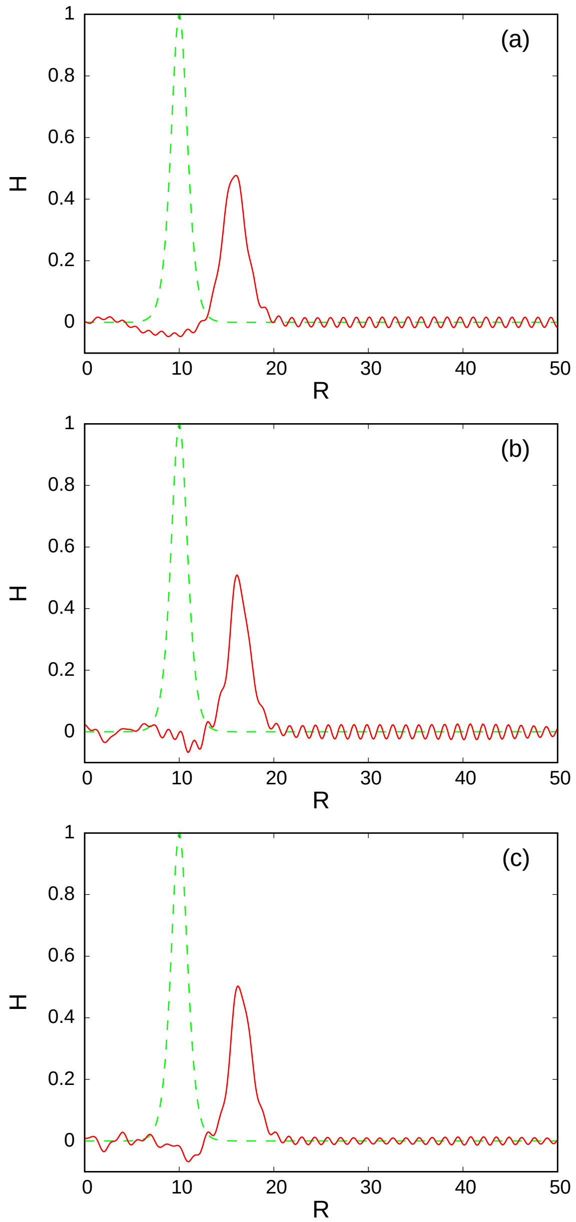

FIG. 2: Numerical solutions of the extended circular KdV equation (17). Green (dotted) line: KdV soliton initial condition at $T = 10$; red (full) line: solution at $T = 30$. (a) Water wave coefficients $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 5/12$ and $c_4 = 19/360$, (b) fifth order derivative only $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ and $c_4 = 19/360$, (c) higher order term HH_{xxx} vanishing, $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 0$ and $c_4 = 19/360$. Here, $a = 1$ for the KdV soliton (42) and $\varepsilon^3 = 0.15$.

VI. UNDULAR BORE RESONANCE

The dependence of the amplitude of the resonant radiation on the coefficients of the higher order terms in the (1+1)D eKdV equation (1) and the ecKdV equation (17) will now be investigated for resonant undular bores. The simplest initial conditions which will generate an undular bore is the step initial condition:

$$u(x,0) = \begin{cases} u_-, & x < x_0, \\ u_+, & x > x_0, \end{cases} \tag{43}$$

at $t = t_0$ for the $(1+1)$D eKdV equation (1), and the step initial condition:

$$H(R,0) = \begin{cases} H_-, & 0 \le R < R_0, \\ H_+, & R > R_0, \end{cases} \tag{44}$$

at $T = T_0$ for the cylindrical eKdV equation (17). For an undular bore to form, we require $u_- > u_+$ and $H_- > H_+$.

Figure 3(a) shows the resonant wavetrain shed by an undular bore for the higher order coefficients c_i, $i = 1,\ldots,4$, taking the water wave values as governed by the eKdV equation. The parameter values are the same as for the solutions displayed in Fig. 1(a) for a resonant water wave solitary wave. As for the resonant solitary wave, the resonant radiation is of very low amplitude for the water wave coefficients, as was noted for resonant flow over topography governed by the eKdV equation with water wave coefficients [19]. This is to be expected as this resonant radiation is determined by matching the velocity of the lead solitary wave of the bore and the phase velocity of linear radiation [30].

However, an undular bore is a modulated wavetrain which evolves from a solitary wave at the leading edge to linear waves at the trailing edge. Each of the component waves of the bore can then resonate, not just the lead solitary wave [32]. This resonance of the entire modulated structure then makes the determination of resonance between radiation and the bore more involved than for a solitary wave. Indeed, it has been found that applying resonance with the lead solitary wave can lead erroneous predictions [58]. The issue of resonance of radiation with an undular bore deserves extensive study. For the case of the evolution of a single solitary wave the shed resonant wavetrain is stable and has an amplitude that is near uniform. However, it is seen that the resonant radiation shed by the undular bore is unstable and that it has a highly modulated amplitude with the radiation in the form of a series of pulses. This is expected as the weakly nonlinear phase of the instability will be governed by an NLS-type equation. This same instability was found for resonant undular bores governed by the Kawahara equation, the eKdV equation with just the higher order fifth derivative [30]. As a simple approximation, we assumed that the radiation profile can be approximated by a sinusoidal curve, and hence the average pulse amplitude is $2/\pi$ times the maximum amplitude. This calculation gives the resonant wave amplitude as 5.1×10^{-3}. For the evolution of a single solitary wave the theory of Section V A gives that the theoretical resonant wave amplitude is 1.6×10^{-3}. While the theoretical prediction does not strictly apply to the scenario of bore evolution since a bore is an extended modulated wavetrain, as discussed above, this is nevertheless a reasonable comparison.

Figure 3(b) displays a bore solution for the Kawahara equation (2). In this case the average amplitude of the

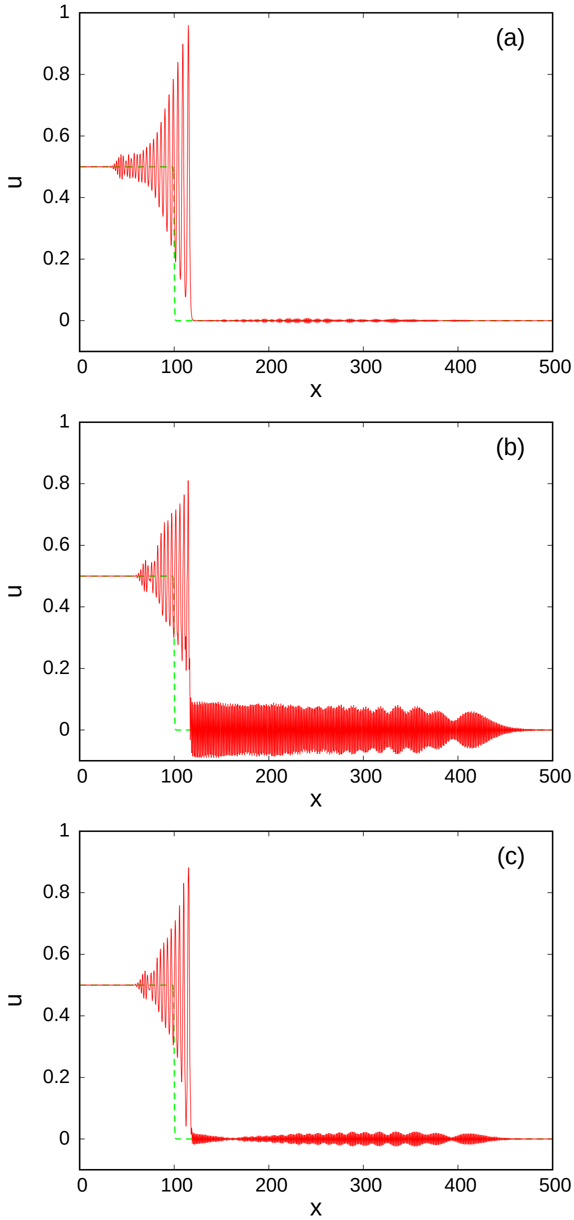

FIG. 3: Numerical solutions of the extended KdV equation (1). Green (dotted) line: step initial condition (43) at $t = t_0 = 10$; red (full) line: solution at $t = 50$. (a) Water wave coefficients $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 5/12$ and $c_4 = 19/360$, (b) fifth order derivative only $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ and $c_4 = 19/360$, (c) higher order term uu_{xxx} vanishing, $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 0$ and $c_4 = 19/360$. Here, $u_- = 0.5$, $u_+ = 0$, $x_0 = 100$ and $\varepsilon = 0.15$.

resonant radiation is 5.4×10^{-2}, while the theoretical amplitude is 4.1×10^{-2}. As for the evolution of a single solitary wave the resonant radiation amplitude is large, which results in a significant decrease in the amplitude of the waves that form the undular bore. Again, the resonant wavetrain is unstable, as is the undular bore itself. The general classification of resonant undular bores terms this undular bore a CDSW, a cross-over dispersive shock wave as it is intermediate between a stable bore and fully resonant bore for which the bore form itself largely disappears as it is shed into the resonant radiation [30]. Figure 3(c) displays a bore solution of the eKdV equation for which only the coefficient of the higher order term $u_x u_{xx}$ vanishes. In this case, the numerical resonant wave amplitude is 1.5×10^{-2}, while the theoretical tail amplitude is 1.3×10^{-2}, again an excellent comparison given that an undular bore is not solely a solitary wave.

In summary, the results for these three examples of undular bore evolution show a qualitatively similar picture to that for the evolution of single solitary waves. For the water wave case the amplitude is an order of magnitude smaller than for the Kawahara equation, while the example with the uu_{xxx} term absent presents an intermediate case. The theoretical predictions based on resonant solitary wave theory are very good, despite this theory not being directly applicable to the evolution of an undular bore.

The dependence of the details of the resonant radiation for resonant undular bores governed by the ecKdV equation (17) will now be investigated. The equivalent circular bore solutions to those of Fig. 4 are shown in Fig. 3. The dependence of the amplitude of the resonant radiation is the broadly consistent with the one dimensional eKdV equation, with the resonant wavetrain being unstable. However, particularly for the cylindrical Kawahara equation, the amplitudes of both the resonant radiation and the undular bore itself are reduced over the equivalent eKdV case due to the decay term $H/(2T)$ in the ceKdV equation. In addition, the undular bore solution of Fig. 4(c) shows that the resonant radiation for the ecKdV equation with the $H_R H_{RR}$ term missing has a reduced amplitude over that for the full ecKdV equation with water wave coefficients, as was the case for the resonant solitary wave solution of the eKdV equation with the $u_x u_{xx}$ term missing displayed in Fig. 2(c). This is in contrast to the one dimensional eKdV equation. The reason for this difference between solutions of the one dimensional and circular equations deserves study. Finally, as for the ecKdV resonant radiation, the addition of $(T - T_0)^{-1/2}$ decay into the theory of subsection V A does not predict the amplitude of the resonant radiation for these bore solutions.

VII. A CONNECTION WITH NONLINEAR OPTICS

The above derivations of weakly nonlinear, long wave equations from the water wave equations may suggest that these equations and related phenomena are exclusive to the shallow water wave problem. However, here we make an unlikely connection with nonlinear optics. In optics the commonly used model is the NLS equation and its variants. The NLS equation is directly associated with deep water waves [5]. Nevertheless, using the above multiscale expansion methodology, we will asymp-

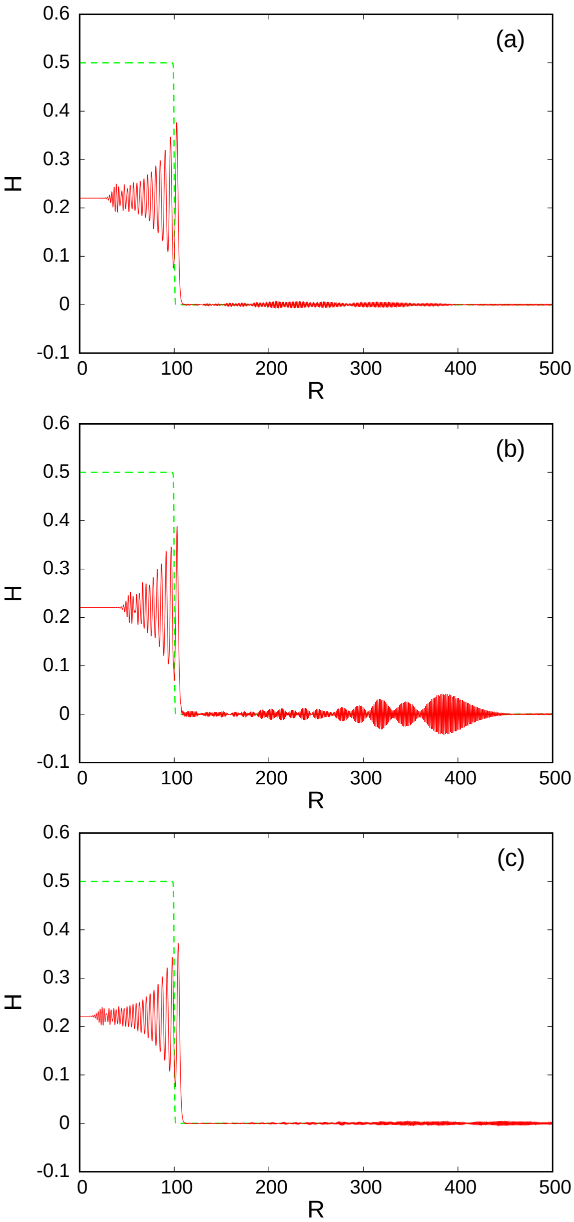

FIG. 4: Numerical solutions of the extended cylindrical KdV equation (17). Green (dotted) line: step initial condition (44) at $T = T_0 = 10$; red (full) line: solution at $T = 50$. (a) Water wave coefficients $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 5/12$ and $c_4 = 19/360$, (b) fifth order derivative only $c_1 = 0$, $c_2 = 0$, $c_3 = 0$ and $c_4 = 19/360$, (c) higher order term HH_{xxx} vanishing, $c_1 = -3/8$, $c_2 = 23/24$, $c_3 = 0$ and $c_4 = 19/360$. Here, $H_- = 0.5$, $H_+ = 0$, $R_0 = 100$ and $\varepsilon^3 = 0.15$.

totically reduce a nonlocal variant of the NLS equation to an appropriate ecKdV equation, thus suggesting that the results of this paper may also find applications in optics.

We will thus consider a prototypical NLS model that governs optical beam propagation in nonlocal, nonlinear media, such as nematic liquid crystals [46, 47, 59] and thermal optical media [60]. In normalized form, this model reads [45, 46, 60]:

$$iu_z + \frac{1}{2}\nabla^2 u - 2\theta u = 0, \tag{45a}$$

$$\nu\nabla^2\theta - 2q\theta = -2|u|^2. \tag{45b}$$

Here, $u = u(x, y, z)$ is the complex electric field envelope of the optical beam propagating in the medium, which evolves along the z-direction, and $\nabla^2 \equiv \partial_x^2 + \partial_y^2$ is the transverse 2D Laplacian. In the context of nematic liquid crystals, the real function $\theta = \theta(x, y, z)$ is the optically induced rotation of the molecular optical axis from its static value in the absence of the light beam, $\nu > 0$ measures the strength of the response of the nematic in space (with a highly nonlocal response corresponding to large ν) and the parameter $q > 0$ is related to the square of the applied, external static electric field which pre-tilts the nematic dielectric [46, 47, 59]. For thermal optical media, θ is the temperature of the medium. It should be noted that a nematic is a focussing medium, so that the nonlinear term $2\theta u$ in (45a) has a positive coefficient. A nematic can be made a defocusing medium through the addition of azo dyes which alters the medium response through the order parameter [61]. In contrast, thermal optical media are typically defoccusing [60].

We now seek solutions of Eqs. (45) with radial symmetry, depending only on the radius r. In this case, Eqs. (45) take the form:

$$iu_z + \frac{1}{2}\left(u_{rr} + \frac{1}{r}u_r\right) - 2\theta u = 0, \tag{46}$$

$$\nu\left(\theta_{rr} + \frac{1}{r}\theta_r\right) - 2q\theta = -2|u|^2. \tag{47}$$

We now introduce the Madelung transformation:

$$u = u_0\sqrt{\rho}e^{i\phi}, \tag{48}$$

where $\rho = \rho(r, z)$ and $\phi = \phi(r, z)$ denote the density and phase of the field u, and $u_0 \in \mathbb{R}$ is a constant. On substituting the polar form (48) into Eqs. (46)–(47), and separating real and imaginary parts, we obtain the system:

$$-2r\theta\rho^2 - r\rho^2\phi_z + \frac{1}{4}\rho\rho_r - \frac{1}{8}r\rho_r^2 - \frac{1}{2}r\rho^2\phi_r^2 + \frac{1}{4}r\rho\rho_{rr} = 0, \tag{49}$$

$$r\rho_z + \rho\phi_r + r\rho_r\phi_r + r\rho\phi_{rr} = 0, \tag{50}$$

$$\nu\left(\theta_{rr} + \frac{1}{r}\theta_r\right) - 2q\theta = -2u_0^2\rho. \tag{51}$$

Next, we seek solutions of this system in the form of the asymptotic expansions:

$$\rho = 1 + \varepsilon\rho_1 + \varepsilon^2\rho_2 + \varepsilon^3\rho_3 + \cdots, \tag{52}$$

$$\phi = -\frac{2u_0^2}{q}z + \varepsilon^{1/2}\phi_1 + \varepsilon^{3/2}\phi_2 + \varepsilon^{5/2}\phi_3 + \cdots, \tag{53}$$

$$\theta = \frac{u_0^2}{q} + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \varepsilon^3\theta_3 + \cdots, \tag{54}$$

where the unknown functions ρ_j, ϕ_j and θ_j ($j = 1, 2, 3, \ldots$) now depend on the stretched variables:

$$R = \varepsilon^{1/2}(r - cz), \quad Z = \varepsilon^{3/2} z, \tag{55}$$

with c being an unknown velocity, to be determined through self-consistency. The small parameter ε measures the deviation of the solution from the background level u_0, so that these asymptotic expansions are relevant to small amplitude waves.

Substituting the expansions (52)-(54) into Eqs. (49)-(51), we obtain a set of equations at the different orders in ε. In particular, at the leading order, namely at $O(\varepsilon^{-1/2})$, we derive the following equations:

$$cq\phi_{1R} - 2u_0^2\rho_1 = 0, \quad q\theta_1 - u_0^2\rho_1 = 0, \tag{56}$$

while at $O(1)$ we obtain:

$$\phi_{1RR} - c\rho_{1R} = 0. \tag{57}$$

The compatibility condition of the above linear equations yields $c^2 = 2u_0^2/q$, i.e., c is the so called "speed of sound".

At the next order of approximation, i.e., at $O(\varepsilon^{1/2})$, we obtain the following set of nonlinear equations:

$$\begin{aligned}&c^2Z\phi_{2R} - 2cZ\theta_2 + \frac{1}{4}cZ\rho_{1RR} - 2R\theta_1 - 4cZ\theta_1\rho_1\\ &-cZ\phi_{1Z} + cR\phi_{1R} + 2c^2Z\rho_1\phi_{1R} - \frac{1}{2}cZ\phi_{1R}^2 = 0, \quad (58a)\\ &2cqZ\theta_2 - 2cu_0^2Z\rho_2 + 2qR\theta_1 - 2u_0^2Z\rho_1 - c\nu Z\theta_{1RR} = 0, \quad (58b)\end{aligned}$$

while the equation obtained at $O(\varepsilon)$ reads:

$$\begin{aligned}&cZ\phi_{2RR} - c^2Z\rho_{2R} + cZ\rho_{1Z} - cR\rho_{1R}\\ &+\phi_{1R} + cZ\rho_{1R}\phi_{1R} + R\phi_{1RR} + cZ\rho_1\phi_{1RR} = 0. \quad (59)\end{aligned}$$

We now remove ρ_2, ϕ_2 and θ_2 from the equations at the orders $O(\varepsilon^{1/2})$ and $O(\varepsilon)$. We thus derive the cylindrical KdV (cKdV) equation for the field amplitude ρ_1

$$\rho_{1Z} + \frac{3c}{2}\rho_1\rho_{1R} - \frac{\alpha}{8c}\rho_{1RRR} + \frac{1}{2Z}\rho_1 = 0, \tag{60}$$

where the parameter α is given by $\alpha = 1 - 2c^2\nu/q$. Notice that the above cKdV equation has been used to describe ring dark (for $\alpha > 0$) and antidark (for $\alpha < 0$) solitons in nonlocal, nonlinear media [62, 63].

Proceeding to the higher-order of approximation, we obtain at $O(\varepsilon^{3/2})$ the following equations:

$$\begin{aligned}&2cqZ\theta_3 - 2cu_0^2Z\rho_3 + 2qR\theta_2 - 2u_0^2R\rho_2 - \nu\theta_{1RR}\\ &-\nu R\theta_{1RR} - c\nu Z\theta_{2RR} + c^2Z\phi_{3R} - 2cZ\theta_3 - 2R\theta_2\\ &-4R\theta_1\rho_1 - 4cZ\theta_2\rho_1 - 2cZ\theta_1\rho_1^2 = 0, \quad (61a)\\ &-4cZ\theta_1\rho_2 - R\phi_{1Z} - 2cZ\rho_1\phi_{1Z} - cZ\phi_{2Z} + \frac{1}{4}\rho_{1R}\\ &-\frac{1}{8}cZ\rho_{1R}^2 + 2cR\rho_1\phi_{1R} + c^2Z\rho_1^2\phi_{1R} + 2c^2Z\rho_2\phi_{1R}\\ &-\frac{1}{2}R\phi_{1R}^2 - cZ\rho_1\phi_{1R}^2 + cR\phi_{2R} + 2c^2Z\rho_1\phi_{2R}\\ &-cZ\phi_{1R}\phi_{2R} + \frac{1}{4}R\rho_{1RR} + \frac{c}{4}Z\rho_1\rho_{1RR} + \frac{c}{4}Z\rho_{2RR} = 0, \quad (61b)\end{aligned}$$

and at $O(\varepsilon^2)$ the equation:

$$\begin{aligned}&R\rho_{1Z} + cZ\rho_{2Z} - cR\rho_{2R} - c^2Z\rho_{3R} + \rho_1\phi_{1R}\\ &+R\rho_{1R}\phi_{1R} + cZ\rho_{2R}\phi_{1R} + \phi_{2R} + cZ\rho_{1R}\phi_{2R}\\ &+R\rho_1\phi_{1RR} + cZ\rho_2\phi_{1RR} + R\phi_{2RR}\\ &+cZ\rho_1\phi_{2RR} + cZ\phi_{3RR} = 0. \quad (62)\end{aligned}$$

We now remove the fields ρ_3, ϕ_3 and θ_3, and employ the equations obtained at the previous orders to express the fields $\theta_{1,2}$ and $\phi_{1,2}$ in terms of the amplitudes ρ_1 and ρ_2. In this manner, we obtain the equation:

$$\begin{aligned}&\frac{c^2q + 2u_0^2}{cq}\rho_{2Z} + \frac{\left[cq\left(-c^2q + 2u_0^2\right)R + 4u_0^2\left(c^2q + u_0^2\right)Z\rho_1\right]}{c^2q^2Z}\rho_{2R} + \frac{-c^2q + 2u_0^2(1-\alpha)}{4c^2q}\rho_{2RRR}\\ &+\frac{2u_0^2\left(cq + 2\left(c^2q + u_0^2\right)Z\rho_{1R}\right)}{c^2q^2Z}\rho_2\frac{\left[\left(cq\left(-3c^4q^2 + 2c^2qu_0^2 + 8u_0^4\right)R + 6u_0^4\left(-3c^2q + 4u_0^2\right)Z\rho_1\right)\rho_1\right]}{2c^4q^3Z}\rho_{1R}\\ &+\frac{u_0^2}{2c^2qZ^2}\partial_R^{-1}\rho_1 - \frac{\left(c^2q + 2u_0^2\right)R}{2c^2qZ^2}\rho_1 - \frac{2cqu_0^2\left(c^2q + 2u_0^2\right)}{2c^4q^3Z}\partial_R^{-1}\rho_1\rho_{1R} + \frac{\left(7c^2q - 8u_0^2\right)u_0^2}{4c^3q^2Z}\rho_1^2\\ &+\frac{11c^4q^2 + 4u_0^4(2-\alpha) - c^2qu_0^2(18 - 7\alpha)}{8c^4q^2}\rho_{1R}\rho_{1RR} - \frac{3c^2q - 2u_0^2(1-\alpha)}{8c^3qZ}\rho_{1RR}\\ &+\frac{3c^4q^2 + 8u_0^4 - 2c^2qu_0^2(3+\alpha)}{8c^4q^2}\rho_1\rho_{1RRR} + \frac{cq - c^2qR(2-\alpha) + 2u_0^2(2 + 2R - 2\alpha - R\alpha)}{8c^4q^2Z}\rho_{1RRR}\\ &+\frac{-c^2q\alpha + u_0^2\left(4 - 6\alpha + 3\alpha^2\right)}{32c^4q}\rho_{1RRRRR} = 0\end{aligned} \tag{63}$$

To simplify this higher order equation, we multiply Eq. (63) by ε, and add it to the cKdV equation Eq. (60). We then introduce the combined amplitude function:

$$Q = \rho_1 + \varepsilon\rho_2,$$

and obtain the extended cKdV equation for the field $Q(R, Z)$:

$$Q_Z + \frac{3c}{2}QQ_R - \frac{\alpha}{8c}Q_{RRR} + \frac{1}{2Z}Q + \varepsilon\left(-\frac{3c}{8}Q^2Q_R + \frac{8+5\alpha}{32c}Q_RQ_{RR} + \frac{\alpha-2}{16c}QQ_{RRR} + \frac{4-8\alpha+3\alpha^2}{128c^3}Q_{RRRRR}\right)$$
$$+\frac{\varepsilon}{Z}\left(\frac{3}{16}Q^2 - \frac{3\alpha}{16c^2}Q_{RR} - \frac{1}{2}Q_R\partial_R^{-1}Q\right) + \frac{\varepsilon}{Z^2}\left(-\frac{R}{2c}Q + \frac{1}{8c}\partial_R^{-1}Q\right) = 0. \quad (64)$$

VIII. CONCLUSIONS

In this work, starting from the Euler (or water wave) equations, we have derived the extended cylindrical Korteweg-de Vries (ecKdV) equation in polar coordinates and the extended Kadomtsev-Petviashvili (eKP) equation in Cartesian coordinates. In so doing, all higher order nonlinear, dispersive and nonlinear-dispersive terms at the next order were found and, additionally, an inherited property that only arises in such higher dimensional settings was revealed: both the ecKdV and eKP equations incorporate nonlocal terms that are not present in the $(1 + 1)$D case— i.e., in the extended KdV model.

Furthermore, these higher order corrections were used to examine the resonant radiation generated by solitary wave and undular bore solutions of these extended equations. It was found that the form of the resonant radiation is highly dependent on the coefficients of the higher order nonlinear, dispersive and dispersive/nonlinear terms. While the overall form of the resonant radiation is broadly similar for the one dimensional and circularly symmetric cases, there are some differences which deserve further examination and analysis. In addition, while there is an existing asymptotic theory for resonant solitary waves governed by the (one dimensional) eKdV equation, no such theory exists for circular solitary waves governed by the ecKdV equation, nor for one dimensional or circularly symmetric resonant undular bores. Again, the issue of resonant undular bores deserves further analysis as these also arise for gravity-capillary waves [30].

Finally, borrowing the same asymptotic expansion method, we made a connection between the water wave context with that of nonlinear optics and derive from a nonlocal NLS equation the same extended cylindrical KdV system (with appropriate coefficients). This suggests that phenomena that can be predicted and observed in shallow water may also occur in optics.

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